Algebraic Geometry Example Sheet 4: Lent 2025

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `hk439@cam.ac.uk`. In all questions, k is an algebraically closed field of characteristic 0. A (*) indicates a more difficult question.

1. Let V be the projective closure of the affine curve $y^3 = x^4 + 1$. Verify that this curve is smooth and has a unique point at infinity. Calculate the divisor of the differential $\omega = \frac{dx}{y^2}$.

2. A curve V is covered by two affine pieces (with respect to different embeddings) which are affine plane curves with equations $y^2 = f(x)$ and $v^2 = g(u)$ respectively, with f a square-free polynomial of even degree $2n > 4$ and $u = 1/x, v = y/x^n$ in $k(V)$. Determine the polynomial $g(u)$ and show that the canonical class has degree $2n - 4$. Why can we not just say that V is the zero locus of the homogenization of $y^2 = f(x)$?

3. Let V be a curve and $P \in V$. Prove that there exists a non-constant rational function on V which is regular away from P.

4. Let V be a curve and $P \in V$. Prove that the variety $V \setminus \{P\}$ is affine.

5. (*) Calculate the degree of the canonical divisor to prove the genus-bidegree formula for curves in $\mathbb{P}^1 \times \mathbb{P}^1$: if F is a bihomogeneous polynomial of bidegree (d, e) and $V = Z(F) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve, then the genus of V is $(d-1)(e-1)$. [Try to follow the proof for plane curves, in this case relating the canonical divisor to the horizontal and vertical divisors i.e. fibers of the projection maps].

6. Let V be a curve with $g(V) > 1$ and suppose there exists an effective divisor D of degree 2 on V with $\ell(D) = 2$. Prove that $\varphi_D : V \to \mathbb{P}^1$ has degree 2 with $\varphi_D^*(\infty) = D$.

7. Prove that every genus 2 curve is hyperelliptic. Prove that there exists a hyperelliptic curve of genus g for every genus $g \geq 2$.

8. Let Q_1 and Q_2 be two smooth quadric surfaces in $\mathbb{P}^3$ such that $Q_1 \cap Q_2$ is a smooth curve. Calculate the genus of this curve. [One way to go about this is via the geometry of the Segre embedding].

9. Construct a smooth projective variety S of dimension 2 and a morphism $\pi : S \to \mathbb{P}^1$ such that (i) away from a finite set of points of $\mathbb{P}^1$, the π-preimage is a smooth curve of genus 1, and (ii) there exists a point $q \in \mathbb{P}^1$ such that $\pi^{-1}(q)$ is a singular curve.